Crude TBP data	% wt.	Temperature, °C.
	13.60	130
	15.40	140
	17.10	150
	18.90	160
Cut point	20.28	168.9
	20.50	170
	22.10	180
	23.80	190

n Value	Stripping	4.7
	Rectifying	7.7

The calculation gives the results of the separation of crude oil into two products at a cut point of 168.9°C. and in this particular case some components were distributed between three products, gasoline, naphtha and kerosene. The experimental data in brackets represent combined data for gasoline plus naphtha, and kerosene plus gas oil, so that only the separation at 168.9°C. is considered.

The consistency of the results obtained for the particular crude oils and distillation units discussed in this paper shows that the analytical treatment of TBP data may be used to designate the fractionation efficiency of refinery equipment and to predict the qualities of refinery streams.

The method of analysis was developed specifically for use with Middle East crude oils having the required linear TBP characteristics. Its application to crude oils which, although linear with respect to the TBP/wt. % relation may have very different hydrocarbon component characteristics from Middle East crudes, would require a laboratory check by equilibruim flash experiments. It is possible that the constants used in Equation (3) relating relative volatility to boiling point difference may not be the same for all types of crude oil.

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NOTATION

A = constant in Equation (3), equal to 10.59/273+T, $°C.^{-1}$
C = slope of TBP curve for crude oil, °C./% wt.
n = number of theoretical plates at total reflux
p = component vapor pressure, lb./sq. in. abs.
T = boiling point, °C.
X = molar flow rate, mole/hr.
W = wt. % based on feed
α = relative volatility
θ = $T - T_o$, °C.
ϕ = overlap coefficient, ½ $C(W_B - W_D)$, °C.
ψ = argument of function in Equation (13), equal to $nA\theta$

Subscripts

B = bottoms product
C = cut point
D = distillate
F = feed
i = component i
o = reference component

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Turbulent Flow in Annular Pipes

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An extension to annular conduits of the well-known method of calculating the resistance coefficient for turbulent flow in circular pipes is presented. The method is new in that it does not borrow any information from laminar flow theory; it gives, therefore, a location for the maximum velocity based only on turbulent flow characteristics. Expressions and a few essential graphs are given for the average as well as inner- and outer-wall resistance coefficients for both hydraulically smooth and rough regimes and also for the transitional regime. A comparison with available experimental information for fully developed turbulent flow is presented.

Ever since the theoretical approaches of Prandtl and Kármán led to the successful synthesis of the perplexing experimental information on uniform flow in circular pipes then available, the logarithmic velocity distribution has been regarded by many as a very general law applicable to a rather wide range of situations. That this could be done only as an approximation was obvious from the assumptions and calculations that led to the adoption of the logarithmic velocity distribution for circular pipes (*1*). Applications have been made to different forms of the cross section (*2*) for which there is no assurance that the secondary flow is negligible. More favorable cases are provided by cross sections of large aspect ratio with gradual variation of transverse dimension, studied recently by the senior author (3). But there is still a form which seems to be most appropriate for an extension of the range of applicability of the logarithmic velocity law; such a form is the concentric annulus, which possesses an axial symmetry akin to that of the circle. In addition, the annular pipe is of practical interest in mechanical, nuclear, and chemical engineering, and has been studied by many researchers during this century, although mostly for rather short conduits. Investigations of flow in long annular conduits are rather scarce, the first (4) of this type having been overlooked for a long time in spite of its being summarized since 1934 in one of the handbooks on flow in pipes (5). To the authors' knowledge, no attempt has ever been made to apply consistently the Prandtl-Kármán approach to the annular pipe, although some authors came very close to it (6, 7). In this paper a calculation of the resistance coefficient for uniform flow in annular pipes is carried out assuming that for fully developed turbulent flow the logarithmic velocity distribution is applicable to the inner and outer layers separated by the isovel surface of maximum velocity and without turbulent action on each other. No assumption is made concerning the location of such a surface or the ratio of wall shearing stresses, both being determinable from the basic assumptions on velocity distribution and lack of interaction, the boundary conditions, and the dynamic equation for the mean flow.

BASIC ASSUMPTIONS AND EXPRESSIONS

Previous attempts to provide a rational treatment for the flow in annular pipes have been based on transferring, with or without corrections, some of the characteristics of laminar flow to turbulent flow; it was assumed either that the location of the maximum velocity is the same for these two regimes of flow (8), or that the ratio of shearing stresses at the walls remains the same for both regimes (9). In fact, the long-ignored work by Lorenz (4) indicates the opposite. Because there is conflicting evidence, and mainly because it appears unjustified to rely on properties of laminar solutions for other than qualitative approximate conclusions concerning their turbulent counterpart, the authors propose a method based exclusively on present knowledge of turbulent flow in pipes. It seems most plausible that the solution for flow in annular pipes should find a place between the two known solutions of turbulent flow in circular conduits and in rectangular conduits of infinite aspect ratio (10). Obviously, the degree of approximation should be of the same order, and comparison with experimental observations of fully established turbulent flow would still be needed. This should be perfectly clear after considering the following basic elements of the authors' approach.

In examining the solution for laminar flow in annular pipes, it is immediately evident that the velocity profile is somewhat asymmetric with respect to the line of maximum velocity. This property should also exist in turbulent flow, but the fact that eddy viscosity cannot be uniform across the flow definitely indicates that the location of the velocity maximum should be different for the two regimes. Thus, it appears better to make assumptions on velocity distribution and lack of turbulent interaction and then proceed to draw conclusions on the characteristics of the flow, as has been done for other cross-sectional forms. The flow in the annular pipe will be divided for this purpose into two layers separated by the cylinder of maximum velocity. The layer around the core or inner pipe of radius r_1 (Figure 1) will be called the inner layer and the layer between the outer pipe of radius r_2 and the cylinder of

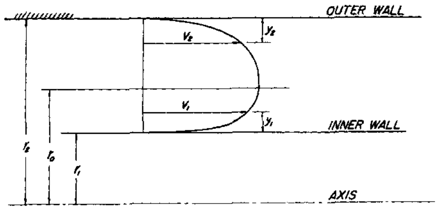

Fig. 1. Definition sketch for flow in annular pipes.

maximum velocity will be designated as the outer layer. It will be accepted that, to the same degree of approximation that applies to circular conduits and to conduits with rectangular cross section of infinite aspect ratio, the logarithmic law can be applied to these inner and outer layers. It seems likely that the two layers may present a logarithmic velocity distribution in the way and to the degree that Schlichting verified experimentally for the asymmetric velocity distribution in his device to test commercial surfaces (11). In fact, the shear-stress distribution in annular pipes departs from the linear form typical of circular and infinitely wide rectangular conduits; the analyses of Kármán and Prandtl have shown, however, that the shear-stress distribution may be varied to a certain extent without affecting notably the final expression for the velocity distribution and the resistance coefficient. Utilizing Kármán's approach, one could take into account the actual shear-stress distribution, but calculations would then be very clumsy and it is not apparent that they would provide a better solution. On the other hand they would not yield expressions that could be compared with the customary form of the resistance coefficient, because no closed form of the integrals involved would then exist. It appears that, for the time being at least, it is more useful to adopt Prandtl's logarithmic velocity distribution as a good enough approximation for the essential purpose of this work, which is the derivation of expressions for resistance coefficients.

Measurements of the velocity distribution in rectangular conduits of large aspect ratio by Laufer (12) indicate a logarithmic velocity distribution. There exist, however, doubts about the constants being exactly the same as in circular pipes, but it is hard to decide whether or not differences are due to the never atttainable condition of truly two-dimensional flow. In fact, an annular section of very large aspect ratio should provide a better experimental case of two-dimensional flow than the corresponding rectangular section. Another circumstance that deserves mention in this analysis of the validity of assumptions to be adopted is that the velocity distribution for laminar flow in annular pipes can be considered only approximately as made up of two parabolas, one for each layer. By analogy, one should realize that the true turbulent velocity law for annular pipes must also deviate somewhat from the logarithmic that should be applicable exactly only to the extreme cases of circular section and annular section of infinite aspect ratio.

In order to introduce certain general concepts, the assumed velocity distributions will symbolically be expressed in the following way for the inner and outer layers, respectively:

$$\frac{v_1}{\sqrt{\tau_1/\rho}} = F_1\left(\frac{y_1}{y_1'}\right) \qquad \text{for } r_1 \leqq r \leqq r_0$$

$$\frac{v_2}{\sqrt{\tau_2/\rho}} = F_2\left(\frac{y_2}{y_2'}\right) \qquad \text{for } r_0 \leqq r \leqq r_2 \qquad (1)$$

At $r = r_0$, $v_1 = v_2$; therefore

$$\frac{\tau_1}{\tau_2} = \frac{F_2^2\left(\frac{r_2 - r_0}{y_2'}\right)}{F_1^2\left(\frac{r_0 - r_1}{y_1'}\right)} \quad (2)$$

is obtained for the ratio of absolute values of shear stresses at the walls. An additional relationship involving these stresses results from the assumption of lack of turbulent interaction and dynamic equations for fluid annular cylinders of unit length, one for each layer:

$$K\pi\,(r_0^2 - r_1^2) = 2\pi r_1 \tau_1$$
$$K\pi\,(r_2^2 - r_0^2) = 2\pi r_2 \tau_2 \quad (3)$$

It has been assumed that the shear stress at $r = r_0$ is zero. Eliminating K between these two expressions, one finds the ratio of wall shear stresses also to be given by

$$\frac{\tau_1}{\tau_2} = \frac{r_0^2 - r_1^2}{r_2^2 - r_0^2}\,\frac{r_2}{r_1} \quad (4)$$

From Equations (2) and (4) one obtains

$$\frac{r_2}{r_1}\,\frac{r_0^2 - r_1^2}{r_2^2 - r_0^2} - \frac{F_2^2}{F_1^2} = 0 \quad (5)$$

which determines the radius r_0 of the isovel of maximum velocity, and consequently the thicknesses $r_0 - r_1$ and $r_2 - r_0$ of the inner and outer layers. In principle, at least, it is conceivable that F_1 and F_2 could be functions of different types representing different regimes in each layer, but in practice they will be functions of similar, if not the same, types. The laminar sublayers will be excluded from the present calculation for turbulent flow in annular pipes, as is also the practice in obtaining the resistance coefficient in circular pipes. Therefore, the functions F_1 and F_2 will simply be logarithmic. It should be noticed that Equation (5), in spite of what the symbolic expressions given to F_1 and F_2 in Equation (1) may suggest, is not necessarily a purely geometric relationship, because for hydraulically smooth walls the parameters y'_1 and y'_2 depend not only on linear dimensions but also on fluid properties.

After r_0 has been calculated, the way is open to determine the overall resistance coefficient f and to compute the local resistance coefficients f_1 and f_2 for the inner and outer walls. Although these two coefficients may be of little interest in hydraulic engineering problems, they are considered to provide valuable although not exact information in problems of transfer of heat and matter (6).

A few more expressions of general validity will be necessary. The average value of the shear stresses of the walls is given by

$$\tau_a = \frac{r_1\tau_1 + r_2\tau_2}{r_1 + r_2} \quad (6)$$

In terms of the pressure gradient, τ_a is also given by

$$\tau_a = KR \quad (7)$$

in which

$$R = (r_2 - r_1)/2 \quad (8)$$

is the overall hydraulic radius.

The resistance coefficients f, f_1, and f_2 are defined by means of

$$\tau_a = \rho f \frac{V^2}{8}; \quad \tau_1 = \rho f_1 \frac{V_1^2}{8}; \quad \tau_2 = \rho f_2 \frac{V_2^2}{8} \quad (9)$$

in which V, V_1, and V_2 indicate average velocities for the whole stream and for its inner and outer layers, respectively. The average velocity V is related to the rate of flow Q by the well-known expression

$$Q = \int_{r_1}^{r_2} 2\pi r v dr = \pi(r_2^2 - r_1^2)\,V \quad (10)$$

Entirely similar expressions serve to define V_1 and V_2.

The hydraulic radii for the inner and outer layers are

$$R_1 = (r_0^2 - r_1^2)/2r_1; \quad R_2 = (r_2^2 - r_0^2)/2r_2 \quad (11)$$

and the Reynolds numbers for the whole stream and its parts will be based on the general definition

$$N_{Re_i} = (4R_iV_i)/\nu \quad (12)$$

in which i may be either suppressed [see Equation (8)] or made equal to 1 or 2.

HYDRAULICALLY SMOOTH FLOW

If both the inner and outer streams function as hydraulically smooth, the velocity distribution may be assumed to be

$$\frac{v_i}{\sqrt{\tau_i/\rho}} = A \log\left(\frac{\sqrt{\tau_i/\rho}}{\nu}\,y_i\right) + B; \quad i = 1, 2 \quad (13)$$

which can also be expressed as

$$\frac{v_i}{\sqrt{\tau_i/\rho}} = A \log\left(C\,\frac{\sqrt{\tau_i/\rho}}{\nu}\,y_i\right) \quad (14)$$

The equation for r_0 takes now the special form:

$$\left(\frac{r_2}{r_1}\,\frac{r_0^2 - r_1^2}{r_2^2 - r_0^2}\right)^{1/2} = \frac{\log\left[\frac{C^2\,\tau_a}{\nu^2\rho}\,\frac{r_2^2 - r_0^2}{2r_2(r_2 - r_1)}\right]}{\log\left[\frac{C^2\,\tau_a}{\nu^2\rho}\,\frac{r_0^2 - r_1^2}{2r_1(r_2 - r_1)}\right]} \quad (15)$$

A dimensionless form can easily be obtained in terms of the average Reynolds number, the average resistance coefficient, and the two ratios:

$$\alpha = r_1/r_2; \quad \beta = r_0/r_2 \quad (16)$$

It is only necessary to introduce the expression for τ_a, as well as the expressions (16) and to regroup terms conveniently; the result will be

$$\left[\frac{\beta^2 - \alpha^2}{\alpha(1 - \beta^2)}\right]^{1/2} = \frac{\log\left[\frac{C}{4\sqrt{2}}\left(\frac{1 - \beta^2}{1 - \alpha}\right)^{1/2}\left(\frac{1 - \beta}{1 - \alpha}\right)N_{Re}\sqrt{f}\right]}{\log\left[\frac{C}{4\sqrt{2}}\left(\frac{\beta^2 - \alpha^2}{\alpha(1 - \alpha)}\right)^{1/2}\left(\frac{\beta - \alpha}{1 - \alpha}\right)N_{Re}\sqrt{f}\right]} \quad (17)$$

Using the general expression for the rate of flow Q as an integral of the velocity over the cross section, one obtains

$$Q = \sqrt{\tau_1/\rho}\,A \int_{r_1}^{r_0} \log\left[C\,\frac{\sqrt{\tau_1/\rho}}{\nu}\,(r - r_1)\right] 2\pi r dr + \sqrt{\tau_2/\rho}\,A \int_{r_0}^{r_2} \log\left[C\,\frac{\sqrt{\tau_2/\rho}}{\nu}\,(r_2 - r)\right] 2\pi r dr \quad (18)$$

Calculation of this integral poses no new problem with respect to the known calculation for the circular section (13). After the integral is obtained, the resulting expression is divided by the cross-sectional area, the expression for V given by the first of Equations (9) is used, and the following dimensionless expression for the resistance coefficient is found:

$$\frac{1}{\sqrt{f}} = \frac{A}{4\sqrt{2}}\left[\frac{1 - \beta^2}{1 - \alpha}\right]^{1/2} \left\{ 2 \log\left[\frac{C}{4\sqrt{2}}\left(\frac{1 - \beta^2}{1 - \alpha}\right)^{1/2}\right.\right.$$

$$\left(\frac{1-\beta}{1-\alpha}\right) N_{Re}\sqrt{f}\Bigg] - \left[\frac{\beta^2-\alpha^2}{\alpha(1-\beta^2)}\right]^{1/2} \frac{\beta-\alpha}{1-\alpha^2}$$

$$(3\alpha+\beta) - \frac{1-\beta}{1-\alpha^2}(3+\beta)\Bigg\} \quad (19)$$

In spite of its complexity, this expression is still of the form

$$\frac{1}{\sqrt{f}} = M_s \log N_{Re}\sqrt{f} + L_s \quad (20)$$

which applies to circular pipes. In fact, Equation (19) reduces to the logarithmic expression for circular pipes if α and β are made vanishingly small. The symbols M_s and L_s have obvious expressions, and their subscript has been selected to indicate smooth flow. If a value of the dimensionless group $N_{Re}\sqrt{f}$ is given, Equation (17) will serve to calculate β as a function of α, and then the functions M_s and L_s can also be calculated, and finally the value of f. In this form, the curves of Figure 2 were determined, as well as those of Figures 3 and 4. The curves in Figure 2 give the ratio β in terms of α and $N_{Re}\sqrt{f}$; these curves illustrate the influence of the geometry and of the flow characteristics on the location of the velocity maximum. In Figure 3 the graphs for the functions M_s and L_s are given; they show a small variation for the first and a more important variation for the second. It should be noticed that M_s takes the same value for $\alpha = 0$ and $\alpha = 1$. The near constancy of M_s in the present case should not lead the reader to the conclusion that M_s is practically constant in all cases. It has been shown elsewhere (3) that it may vary noticeably with the shape of the cross section; in fact it would be enough merely to shift off center the cylinders that form the annular conduit to have important variations of M_s. Figure 4 gives the resistance coefficient in terms of N_{Re} and α.

From the expression for the rate of flow of the inner layer, the following expression for the resistance coefficient f_1 can be obtained:

$$\frac{1}{\sqrt{f_1}} = \frac{A}{4\sqrt{2}}\left\{2\log\left[\frac{C}{4\sqrt{2}}\left(\frac{\beta^2-\alpha^2}{\alpha(1-\alpha)}\right)^{1/2}\right.\right.$$

$$\left.\left.\left(\frac{\beta-\alpha}{1-\alpha}\right)N_{Re}\sqrt{f}\right] - \frac{\beta+3\alpha}{\beta+\alpha}\right\} \quad (21)$$

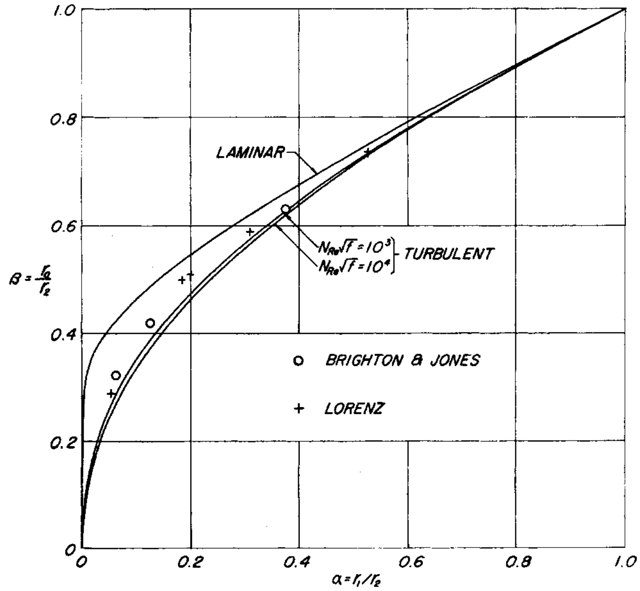

Fig. 2. Relative location of the velocity maximum vs. ratio of inner and outer radii for laminar and hydraulically smooth flow.

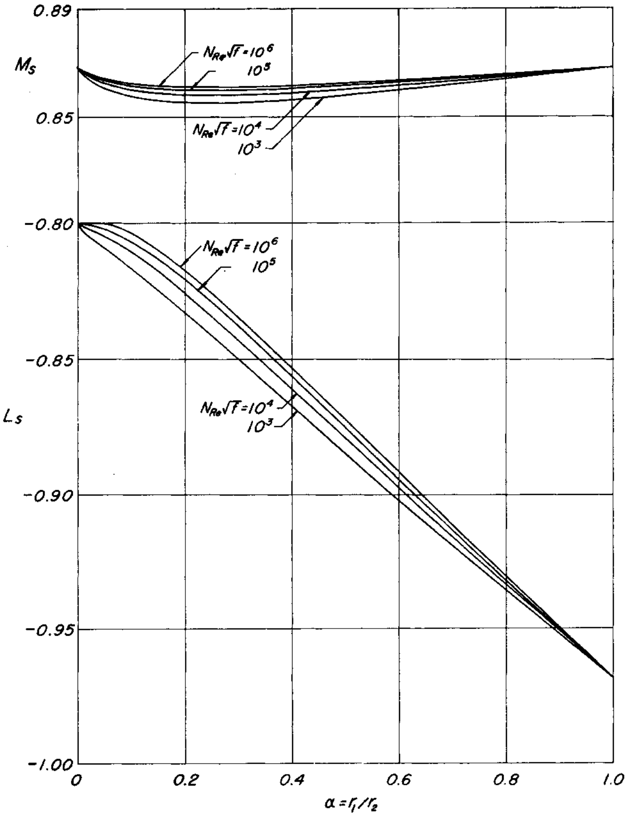

Fig. 3. Shape functions M_s and L_s for hydraulically smooth flow.

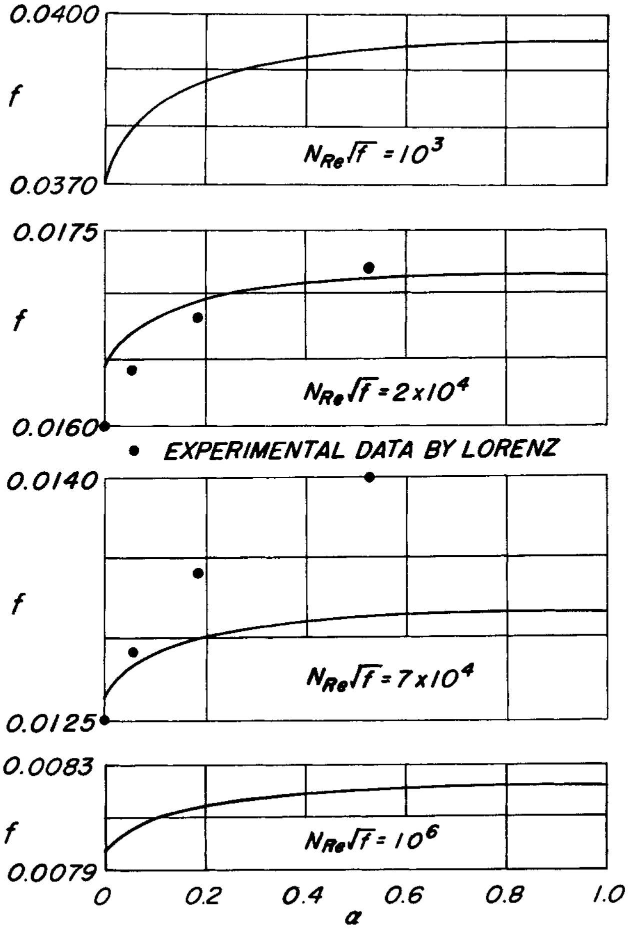

Fig. 4. Average resistance coefficient for hydraulically smooth flow (experimental points by F. R. Lorenz).

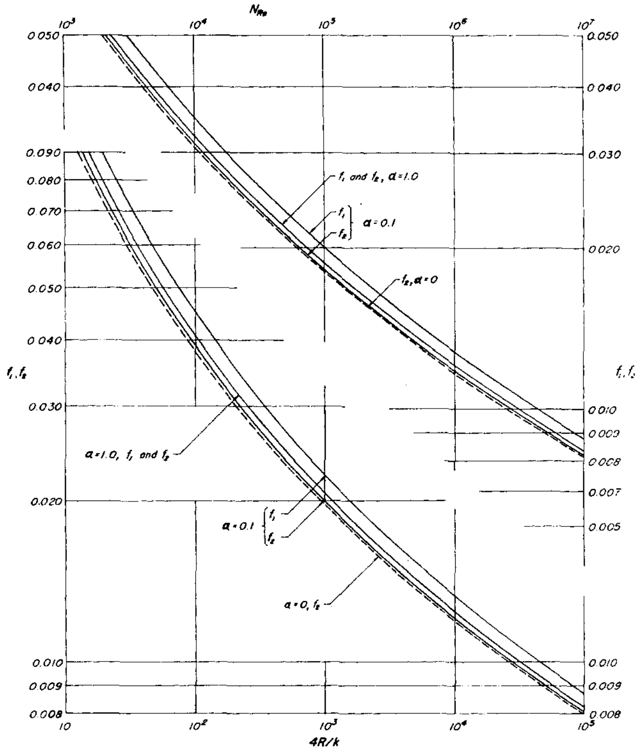

Fig. 5. Resistance coefficients of inner and outer walls for hydraulically smooth flow (upper part) and hydraulically rough flow (lower part).

In an entirely similar manner the expression for f_2 is found to be

$$\frac{1}{\sqrt{f_2}} = \frac{A}{4\sqrt{2}}\left\{ 2\log\left[\frac{C}{4\sqrt{2}}\left(\frac{1-\beta^2}{1-\alpha}\right)^{1/2}\left(\frac{1-\beta}{1-\alpha}\right)N_{Re}\sqrt{f}\right] - \frac{3+\beta}{1+\beta}\right\} \quad (22)$$

To calculate f_1 and f_2, the relations between $N_{Re1}\sqrt{f_1}$ and $N_{Re2}\sqrt{f_2}$ and $N_{Re}\sqrt{f}$ were necessary. From Equations (4), (6), and (9), the following relations were obtained:

$$N_{Re1}\sqrt{f_1} = \left[\frac{\beta^2 - \alpha^2}{\alpha(1-\alpha)}\right]^{3/2} N_{Re}\sqrt{f} \quad (23)$$

$$N_{Re2}\sqrt{f_2} = \left[\frac{1-\beta^2}{1-\alpha}\right]^{3/2} N_{Re}\sqrt{f} \quad (24)$$

For a few values of α, graphs of the resistance coefficients f_1 and f_2 for inner and outer walls, respectively, are given in the upper part of Figure 5. When using this figure, it should be realized that the variations of f_1 and f_2 are very small for $0.5 \leq \alpha \leq 1.0$. The values of f_2 do not differ much from those of f; the range of variation of f_1 is actually very wide but its calculated values are probably less and less reliable with decreasing α.

TRANSITIONAL AND HYDRAULICALLY ROUGH FLOWS

It has long been recognized that wall roughness influences turbulent flow only after a certain Reynolds number has been attained. When two walls are present, one should expect that on one this situation could appear later than on the other, thus leading to hydraulically smooth flow near one wall and transitional or even hydraulically rough flow near the other wall. The case of hydraulically rough flow within both inner and outer layers will be investigated first, and then the possibility of a mixed regime will be discussed.

If both walls function as rough, the velocity distribution in each layer can be represented by an expression such as

$$\frac{v_i}{\sqrt{\tau_i/\rho}} = A\log\left(C'\frac{y_i}{k_i}\right); \quad i = 1, 2 \quad (25)$$

The equation for the radius of maximum velocity now becomes

$$\left(\frac{r_2}{r_1}\frac{r_0^2 - r_1^2}{r_2^2 - r_0^2}\right)^{1/2} = \frac{\log\left[\frac{C'}{k_2}(r_2 - r_0)\right]}{\log\left[\frac{C'}{k_1}(r_0 - r_1)\right]} \quad (26)$$

It has been assumed that the absolute roughness is different for the inner and outer walls in order to show that this general case can be approached without any greater difficulty than the case of equal absolute roughness. The value of r_0 is determined now by purely geometrical parameters; if one wants to include among such parameters the hydraulic radius of each layer, or the overall hydraulic radius, it can easily be done, as indicated below for the case $k_1 = k_2 = k$:

$$\left[\frac{\beta^2 - \alpha^2}{\alpha(1-\beta^2)}\right]^{1/2} = \frac{\log\left[\frac{C'}{2}\frac{1-\beta}{1-\alpha}\frac{4R}{k}\right]}{\log\left[\frac{C'}{2}\frac{\beta-\alpha}{1-\alpha}\frac{4R}{k}\right]} \quad (27)$$

Once the relative roughness $k/4R$ and the shape ratio α are given, the ratio β can be determined with this equation. The curves for β in terms of α and $4R/k$ are given in Figure 6.

The expression for the overall resistance coefficient f can be obtained, as in the previous case, by calculating the discharge Q and then by utilizing the definition equation for the resistance coefficient. The following expression has thus been obtained:

$$\frac{1}{\sqrt{f}} = \frac{A}{4\sqrt{2}}\left[\frac{1-\beta^2}{1-\alpha}\right]^{1/2}\left\{ 2\log\left[\frac{C'}{2}\left(\frac{1-\beta}{1-\alpha}\right)\frac{4R}{k}\right]\right.$$

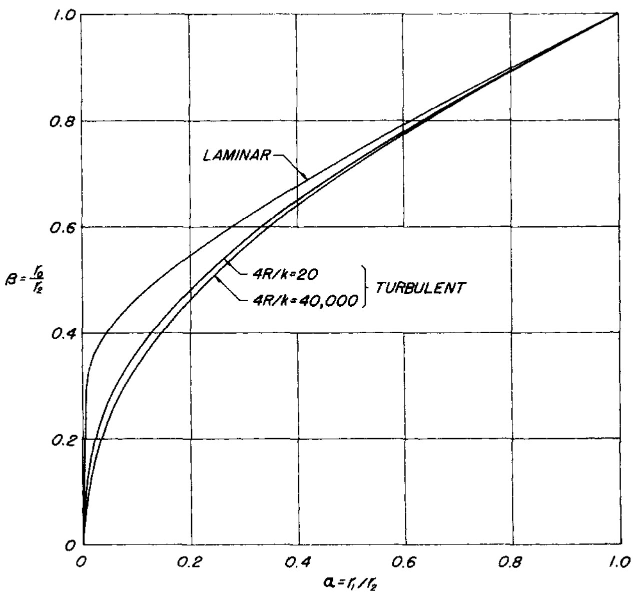

Fig. 6. Relative location of the velocity maximum vs. ratio of radii for hydraulically rough flow.

$$-\left[\frac{\beta^2-\alpha^2}{\alpha(1-\beta^2)}\right]^{1/2}\frac{\beta-\alpha}{1-\alpha^2}(3\alpha+\beta)-\frac{1-\beta}{1-\alpha^2}(3+\beta)\Bigg\} \quad (28)$$

It can be seen that again the expression for the resistance coefficient can be reduced to

$$\frac{1}{\sqrt{f}} = M_r \log\frac{4R}{k} + L_r \quad (29)$$

the logarithmic form being again preserved. The shape functions M_r and L_r for hydraulically rough annular conduits depend on α and $4R/k$; the corresponding curves are given in Figure 7.

The expressions for the inner- and outer-wall resistance coefficients are

$$\frac{1}{\sqrt{f_1}} = \frac{A}{4\sqrt{2}}\left\{2\log\left[\frac{C'}{2}\left(\frac{\beta-\alpha}{1-\alpha}\right)\frac{4R}{k}\right]-\frac{3\alpha+\beta}{\alpha+\beta}\right\} \quad (30)$$

and

$$\frac{1}{\sqrt{f_2}} = \frac{A}{4\sqrt{2}}\left\{2\log\left[\frac{C'}{2}\left(\frac{1-\beta}{1-\alpha}\right)\frac{4R}{k}\right]-\frac{3+\beta}{1+\beta}\right\} \quad (31)$$

In the lower part of Figure 5 curves for f_1 and f_2 illustrate their variation in terms of α and $4R/k$. The remarks on f_1 and f_2 made for hydraulically smooth flow again apply.

Mixed regimes in the annular conduits should be more likely to occur for small values of α than for values approaching 1, when the two layers tend to become equal. It should be reckoned with, in this argument, that the logarithmic velocity distribution is only an approximate one and that the transference of all occurrences found in circular pipes may be affected by unexpected inaccuracies, principally when transition is involved. The authors have performed some calculations to investigate what the behavior in annular conduits would be if there would exist an exact replica of the findings for circular pipes in each of the layers of the flow in annular pipes. The investigation has been based on a roughness function similar to the one given by Schlichting (*14*) to fit the velocity distributions of Nikuradse's experiments for artificially roughened pipes. The function has been varied somewhat to make it consistent with accepted values of the resistance coefficient for circular pipes. This function depends on the wall roughness, and it is conceivable that, if the values of k_1 and k_2 were considerably different, it could be possible to have the smooth regime on one side and the rough regime on the other side. The case that has been chosen for examination in this paper is the one of equal absolute uniform roughness on both walls, because the generality of the analysis is thus not actually lessened—it would be as easy to use any other function than the one selected. It has been considered convenient to provide an approximate information to the design engineer at the same time that a basis for further experimental research is built up.

It was found that mixed regimes occurred only for combinations of transitional flow with either hydraulically smooth or rough flow. Important to the design engineer is to recognize that the case of different wall roughness cannot be treated with an estimate based on some average value of a convenient parameter, and requires calculations entirely similar to the ones presented here for equal roughness. Commercial type of roughness can be considered by adopting in the computations an appropriate form of the roughness function.

Calculations for the transitional flow were performed by trial and error in the following way. For given values of the parameters of the flow the two expressions

$$\frac{\sqrt{\tau_1/\rho}\,k}{\nu} = \frac{N_{Re}\sqrt{f}}{4R/k}\left[\frac{1}{2\sqrt{2}}\left(\frac{\beta^2-\alpha^2}{\alpha(1-\alpha)}\right)^{1/2}\right]$$

$$\frac{\sqrt{\tau_2/\rho}\,k}{\nu} = \frac{N_{Re}\sqrt{f}}{4R/k}\left[\frac{1}{2\sqrt{2}}\left(\frac{1-\beta^2}{1-\alpha}\right)^{1/2}\right] \quad (32)$$

are used to obtain trial values of B_1 and B_2 (hence C_1 and C_2) from the curve given in Figure 8. Then the formula for the ratio β is used:

$$\left[\frac{\beta^2-\alpha^2}{\alpha(1-\beta^2)}\right]^{1/2} = \frac{\log\left[\frac{C_2}{2}\left(\frac{1-\beta}{1-\alpha}\right)\frac{4R}{k}\right]}{\log\left[\frac{C_1}{2}\left(\frac{\beta-\alpha}{1-\alpha}\right)\frac{4R}{k}\right]} \quad (33)$$

To calculate the resistance coefficient, the following expression is applied:

$$\frac{1}{\sqrt{f}} = \frac{A}{4\sqrt{2}}\left[\frac{1-\beta^2}{1-\alpha}\right]^{1/2}\Bigg\{2\log\left[\frac{C_2}{2}\left(\frac{1-\beta}{1-\alpha}\right)\frac{4R}{k}\right]$$
$$-\left[\frac{\beta^2-\alpha^2}{\alpha(1-\beta^2)}\right]^{1/2}\frac{\beta-\alpha}{1-\alpha^2}(3\alpha+\beta)-\frac{1-\beta}{1-\alpha^2}(3+\beta)\Bigg\} \quad (34)$$

The results of the calculations for the average resistance coefficient for fully turbulent flow in annular conduits are summarized in Figure 9.

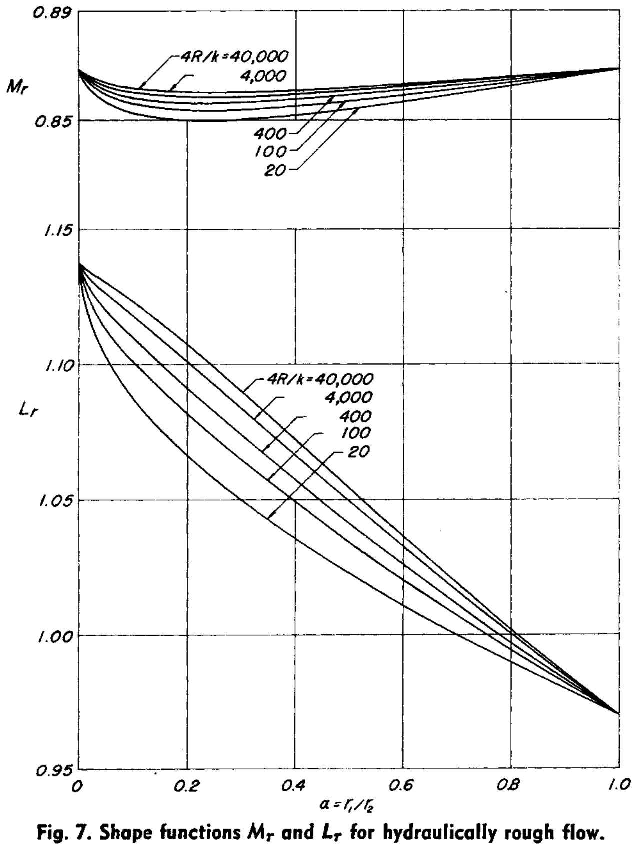

Fig. 7. Shape functions M_r and L_r for hydraulically rough flow.

COMPARISON WITH EXPERIMENTAL RESULTS

In spite of the rather abundant literature on flow through annuli (*15, 16*) there are only few papers on fully established turbulent flow in conduits. Two of them have been selected for purposes of comparison with the calculations for the hydraulically smooth regime, because they show good evidence that precautions were taken to

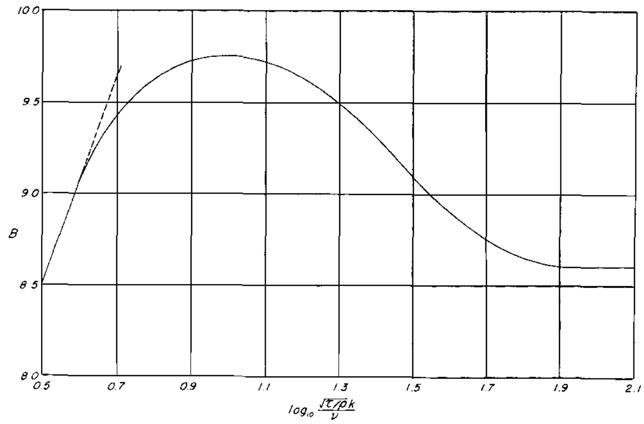

Fig. 8. Modified roughness function.

have well-established flow and because values of the overall resistance coefficient are given in them for both circular and annular cross sections. One of these two papers is Lorenz's of 1932, and the other is the very recent work of Brighton and Jones (*17*). To Lorenz it was obvious that the location of the velocity maximum could not be the same for laminar and turbulent flow, and he gave two different curves for these two regimes. Brighton and Jones also verified that for turbulent flow "the radius of the point of maximum velocity is definitely less than for laminar flow" (*17*). Experimental points for the ratio β in terms of the ratio α, by Lorenz and by Brighton and Jones, have been indicated in Figure 2; a fair amount of scattering should be expected because of the flatness of the velocity curves near the velocity maximum. However, it is obvious from Figure 2 that there is a systematic deviation from the theoretical curves obtained by the present method.

To judge from another point of view the degree of agreement with experimental results involved in assuming logarithmic velocity distribution, this has been compared in Figure 10 with velocity curves of Brighton and Jones. It can be seen that the shapes of assumed and actual velocity curves are different, although their quantitative differences are relatively small. It is interesting to note that the case $\alpha = 0$ (circular section) also presents the type of discrepancy illustrated by Figure 10. Because the differences between assumed and experimental velocity curves are relatively small, it can be expected that overall characteristics like the resistance coefficient should be predicted rather accurately.

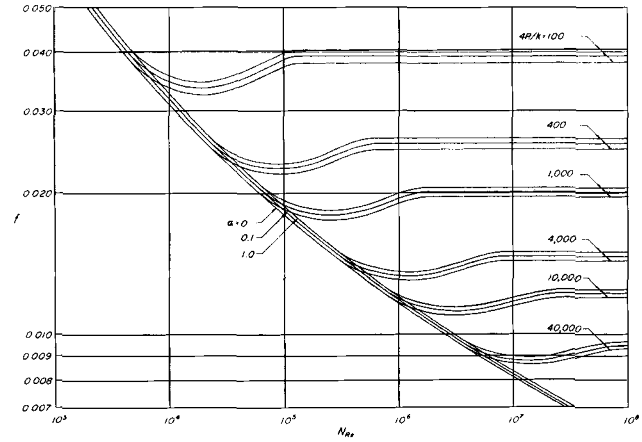

Fig. 9. Average resistance coefficient for turbulent flow in annular pipes with uniform roughness.

Comparison of calculated values with experimental determinations of the average coefficient is presented in Figures 4 and 11. There is a definite discrepancy with the curves given by Lorenz, and much better agreement with those of Brighton and Jones. However, Lorenz's results possess a very well-defined trend of variation of the resistance coefficient as a function of α, also indicated by the present calculations though in lesser degree, while the results of Brighton and Jones do not show such a trend. On one hand, Lorenz's results may not correspond strictly to hydraulically smooth pipes judging by the shape and location of his curves for f as a function of N_{Re} (*4*). On the other hand, the variation indicated theoretically is rather small, and because of unavoidable experimental errors the trend is bound not to show at all unless very high accuracy is obtained. Brighton and Jones have indicated modifications of the constants A and B of the logarithmic velocity-distribution law that would represent better their experimental results for annular pipes, but adoption of their values would leave us with unacceptable deviations with respect to the circular pipes, for which well-established constants are available. In addition, they found that the velocity distribution in the inner layer, for small values of α, departs substantially from the logarithmic law. Until experimental values for f possessing a definite trend in terms of α are obtained, it seems better not to attempt modifications of the constants A and B. Very few experiments for annular rough pipes are available. Besides Lorenz's experiments, which were classified as corresponding to hydraulically smooth flow, although there could be in them a small influence of roughness, mention will be made of the work by Owen (*18*) in 1951 and the much more recent by Nicol and Medwell (*19*). It is difficult to compare Owen's measurements with the present calculations, because no value of k can surely

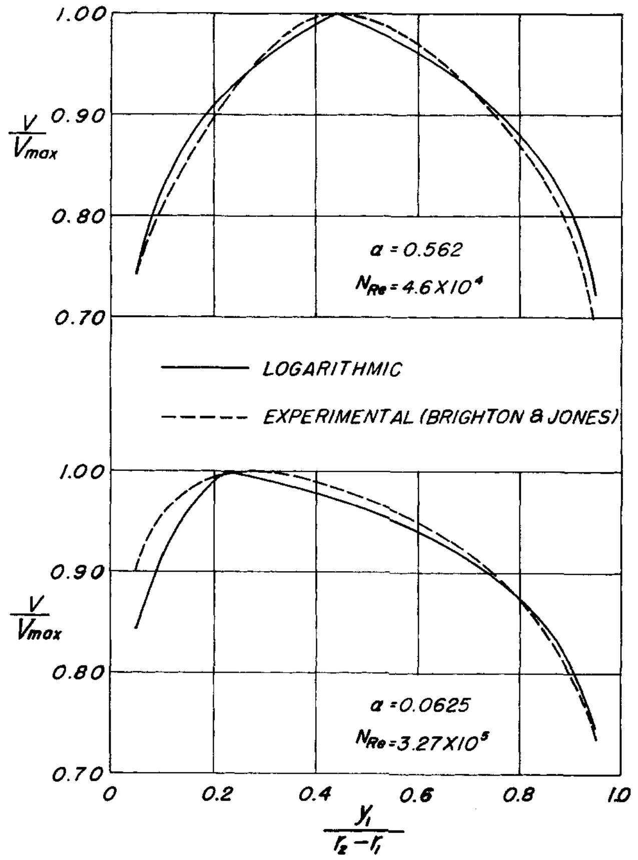

Fig. 10. Calculated and experimental velocity profiles.

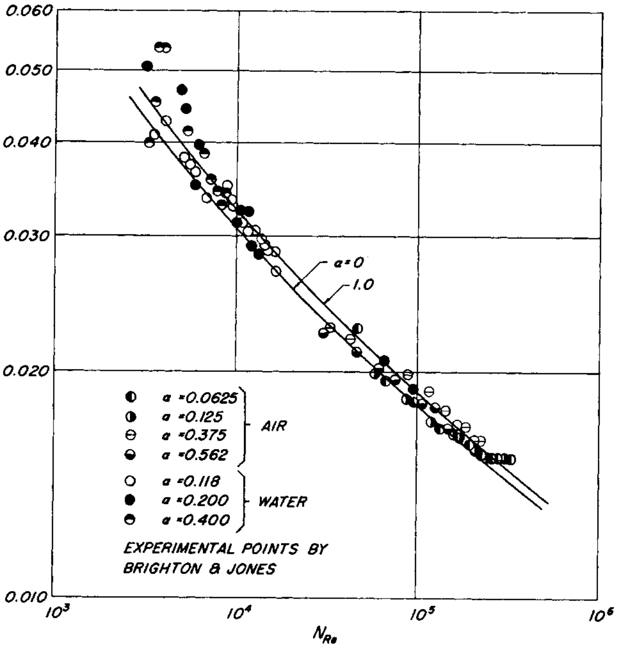

Fig. 11. Calculated and experimental average resistance coefficients for hydraulically smooth flow in annular pipes (experimental results by Brighton and Jones).

be assigned to his pipes; nonetheless, his experiments show a trend of variation for the resistance coefficient in agreement with results given in this paper. Nicol and Medwell experimented with an inner pipe that was artificially roughened while the outer pipe was always smooth. They determined from their measurements a curve which is quite similar to the one given in Figure 8, deviations in the transition region being the expected ones. It can, therefore, be considered that Nicol and Medwell's experiments confirm the usefulness of adapting the Prandtl-Kármán approach to annular pipes.

CONCLUSIONS

The Prandtl-Kármán expression for the resistance coefficient of turbulent flow in circular pipes has been extended to turbulent flow in both smooth and rough annular conduits. Contrary to previous approaches, the present one has not been based on properties of laminar flow, and it provides the location of the maximum velocity independently. Comparison with available experimental results for fully developed turbulent flow in annular conduits shows that the expressions for the resistance coefficients are sufficiently accurate for engineering calculations. The trend of the influence of aspect ratio on resistance, as revealed by the analytical approach, is by and large confirmed by the experiments of Lorenz and Owen; however, the more recent experiments by Brighton and Jones do not indicate any distinguishable trend. From an applied engineering point of view the discrepancies may well be considered small, but from a fundamental point of view they should provide a basis for further investigation—both theoretically and analytically—of the effects of shape on resistance to flow in conduits.

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NOTATION

A = coefficient in logarithmic velocity law
C, C' = coefficients in logarithmic velocity law
f = average resistance coefficient
f_1, f_2 = resistance coefficients for inner and outer walls
k = absolute roughness
k_1, k_2 = absolute roughness on inner and outer walls
K = piezometric gradient
M_r, L_r = shape functions for hydraulically rough flow
M_s, L_s = shape functions for hydraulically smooth flow
Q = discharge
r = radial distance
r_0 = radius at point of maximum velocity
r_1, r_2 = radii of inner and outer walls
R = average hydraulic radius
R_1, R_2 = hydraulic radii for inner and outer layers
N_{Re} = average Reynolds number
$N_{Re1,2}$ = Reynolds numbers for inner and outer layers
v = local velocity
v_1, v_2 = local velocities in inner and outer layers
V = average velocity for whole stream
V_1, V_2 = average velocity for inner and outer layers
y_1, y_2 = distances from inner and outer walls
y_1', y_2' = length parameters of turbulent flow

Greek Letters

α = ratio of inner and outer radii
β = relative distance from outer wall for velocity maximum point
ν = kinematic viscosity
ρ = mass density
τ_a = average shear stress
τ_1, τ_2 = shear stresses on inner and outer walls

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